Equilibrium-restricted solid-on-solid growth model on fractal substrates

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The equilibrium-restricted solid-on-solid growth model on fractal substrates is studied by introducing a fractional Langevin equation. The growth exponent β and the roughness exponent α defined, respectively, by the surface width via $W \sim t^{\beta}$ and the saturated width via $W_{\rm sat} \sim L^{\alpha}$, L being the system size, were obtained by a power-counting analysis, and the scaling relation $2\alpha + d_{\rm f} = z_{\rm RW}$ was found to hold. The numerical simulation data on Sierpinski gasket, checkerboard fractal, and critical percolation cluster were found to agree well with the analytical predictions of the fractional Langevin equation.

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I. INTRODUCTION

Over the last several decades, there has been a considerable number of studies regarding the surface-roughening of equilibrium and nonequilibrium interfaces using various continuum growth equations and discrete atomistic models [1–4]. The surface-roughening phenomenon is associated with a wide variety of other systems such as domain walls in the two-dimensional (2D) random bond Ising model [5], randomly stirred fluids [6], ballistic aggregation [7], and directed polymer in a random potential [8,9].

The surface growth phenomena can be categorized into various universality classes [1–4]. The most prominent and well-known one of them is the Kardar-Parisi-Zhang (KPZ) class, for which the growth can be described by a continuum equation, given as [8]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\vec{\nabla} h)^2 + \eta(\vec{r},t), \tag{1}$$

where η is the Gaussian random variable which satisfies

$$\langle \eta(\vec{r},t)\eta(\vec{r}\,',t') = 2\Gamma\delta(\vec{r}-\vec{r}\,')\delta(t-t')\ , \tag{2}$$

with Γ describing the strength of the noise. The restricted solid-on-solid (RSOS) model for nonequilibrium growth on a regular lattice is believed to be described by the KPZ equation [10]. The dynamic rule of the RSOS growth model is to randomly select a site x on a substrate and then either to deposit or to evaporate a particle with unequal probabilities, $h(x) \to h(x) \pm 1$ (within the solid-on-solid condition), provided that the restriction on the local height difference

$$|\vec{\nabla} h| = |h(x) - h(x')| \leq 1 \tag{3}$$

is obeyed between the selected site and the nearest-neighbor sites. If this RSOS condition is not satisfied, the corresponding deposition or evaporation event is forbidden. No relaxation or hopping of the deposited atom is allowed [10]. However, in regards to the equilibrium RSOS growth with the same probabilities for deposition and evaporation, the nonlinear term vanishes and Eq. (1) is reduced to the Edwards-Wilkinson (EW) equation, given as [11]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(\vec{r},t). \tag{4}$$

The growth models, which can be described by the EW equation, are known to belong to the EW universality class. The other continuum equation is the fourth-order Herring-Mullins (HM) equation, describing surface dynamics under curvature constraints [12],

$$\frac{\partial h(\vec{r},t)}{\partial t} = -\nu_4 \nabla^4 h + \eta(\vec{r},t). \tag{5}$$

Since Eqs. (4) and (5) are linear, one can solve them exactly, and the growth models described by the HM equation are known to belong to the HM universality class.

Since the surface structure of many growth processes is self-affine, most efforts have been concentrated on measuring the surface fluctuations. The surface width W is defined as the standard deviation or, equivalently, the root-mean-square fluctuation of the surface height

$$W(t,L) = \langle [h(\vec{r},t) - \overline{h(t)}]^2 \rangle^{1/2}, \tag{6}$$

where $h(\vec{r},t)$ are the local height variables of the d-dimensional interface, $\overline{h(t)}$ is their spatial average, and $\langle \cdots \rangle$ denotes the average over many samples. Here, d is written as a substrate dimension and, therefore, the total dimension is $d+1$. The interesting quantities in the growth process are the exponents which describe the self-affine surface structure [4,8,11,13]. The scaling hypothesis is such that in a finite system of lateral size L, the mean-square fluctuation W^2 of the surface height starting from a flat substrate scales as [13],

$$W^2(L,t) \sim L^{2\alpha} g(t/L^z) \to t^{2\beta}, \quad t \ll L^z,$$
$$\to L^{2\alpha}, \quad t \gg L^z, \tag{7}$$

where α and β are, respectively, the roughness exponent and the growth exponent, and the ratio of the two defines the

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dynamic exponent, $z=\frac{\alpha}{\beta}$. The roughness exponent α is the quantity that describes the characteristic of the saturated surface fluctuation at a sufficiently late time, and the dynamic exponent z describes the lateral correlation of the surface height with a time-dependent length scale $\xi_x(t)\sim t^{1/z}$.

The "equilibrium" RSOS (ERSOS) growth model [10] on a regular lattice is generally believed to be described by the EW equation which can be solved exactly [11]. On the other hand, the KPZ Eq. (1), describing the nonequilibrium RSOS growth model, is not yet solved except for the one-dimensional case, and most studies on the RSOS model were devoted to Monte Carlo simulations [10,14–18]. There have also been studies regarding both equilibrium and nonequilibrium RSOS models on fractal substrates. Despite several previous studies regarding the growth on various fractal substrates [19,20], there is no simple understanding about the values of the critical exponents. Therefore, the questions that might naturally arise are as follows: how do the growth and roughness exponents β and α depend on the fractal substrates? Is there any continuum equation associated with the discrete models?

In this study, the ERSOS growth model is studied on fractal substrates, in attempting to envision an EW-type discrete model on fractal substrates, a fractional Langevin equation related to the model is introduced, and the critical exponents are calculated by a power-counting method based on the renormalization-group transformation. Extensive Monte Carlo simulations are carried out for the ERSOS models on various fractal substrates, such as the Sierpinski gasket, Sierpinski carpet, and checkerboard fractal, all of which are embedded in two spacial dimensions. It is found that the values of β and α are not consistent with those obtained from the known exact results by a simple substitution of the substrate dimension with the fractal dimension, but completely correspond to the predictions by a power-counting method on the fractional Langevin equation.

II. EQUILIBRIUM RSOS MODEL ON FRACTAL SUBSTRATES

The nonlinear term on the KPZ equation can be controlled by generalizing the RSOS growth model to allow the evaporation of particles within the RSOS condition. The growth rule regarding the model is to randomly select a site on a substrate and to deposit (evaporate) a particle on the site with the probability $p_+(p_-)$, when the height configuration satisfies the restriction in Eq. (3) after the deposition (evaporation) [10]; otherwise, the deposition (evaporation) process is ignored. The strength of the nonlinearity λ can be controlled by adjusting the deposition probability p_+. The lateral growth rate is expected to become proportional to the difference between the evaporation and the deposition rates [10]. Thus, λ should be zero for the equilibrium growth, in which $p_+=p_-$ holds. Therefore, the equilibrium RSOS model on a regular substrate is well described by the EW equation. The growth exponents β and α, characterized by the EW equation, were obtained by a direct integration, yielding $\beta=(2-d)/4$, $\alpha=(2-d)/2$, and $z=2$.

The ERSOS model on a fractal substrate is now considered. The term $\nabla^2 h$ in the EW equation describes the diffusion process and has symmetries under inversion and rotation in the $\vec{r}$ space. On a fractal substrate, it is known that the diffusion is anomalous and the dynamic exponent of random walks z_{RW}, defined by the rms end-to-end distance via $\langle R^2\rangle \sim t^{2/z_{\mathrm{RW}}}$, is larger than 2. Also, no symmetry exists under either inversion or rotation in usual fractal substrates. Therefore, we assume that the term associated with diffusion is $\nabla^{z_{\mathrm{RW}}}h$, and introduce a fractional Langevin equation

$$\frac{\partial h}{\partial t}=\nu\nabla^{z_{\mathrm{RW}}}h+\eta(\vec{r},t). \quad (8)$$

In order to utilize the power-counting approach on the fractional Langevin equation on fractal substrates, it is considered that a size $x'\times x'$ system is rescaled by a factor b into the smaller $x\times x$ system, i.e., $x'\to bx$. Then, in the $t \gg L^z$ limit, the surface width and the evolution time are rescaled, respectively, as $h'=x'^{\alpha}\to(bx)^{\alpha}=b^{\alpha}h$ and $t'\to b^z t$. The noise is distributed over the fractal substrate of $x^{d_{\mathrm{f}}}$ lattice sites during the time t, where d_{f} is the fractal dimension. Therefore, the total noise sum in the volume of $x'^{d_{\mathrm{f}}}t'$ should be rescaled into the noise sum in the volume $x^{d_{\mathrm{f}}}t$. Based on the equation

$$\int\langle\eta(\vec{r}_0,t_0)\,\eta(\vec{r},t)\rangle d^{d_{\mathrm{f}}}r\,dt=2\Gamma, \quad (9)$$

it is clear that the noise η should be rescaled as $\eta'\to b^{-(d_{\mathrm{f}}+z)/2}\eta$ via the central limit theorem. Then, Eq. (8) can be written in its rescaled form as

$$b^{\alpha-z}\frac{\partial h}{\partial t}=\nu b^{\alpha-z_{\mathrm{RW}}}\nabla^{z_{\mathrm{RW}}}h+b^{-(d_{\mathrm{f}}+z)/2}\eta. \quad (10)$$

By comparing the power of each term, the following results are obtained:

$$z=z_{\mathrm{RW}},\quad \alpha=\frac{1}{2}(z_{\mathrm{RW}}-d_{\mathrm{f}}),\quad \beta=\frac{1}{2}\left(1-\frac{d_{\mathrm{f}}}{z_{\mathrm{RW}}}\right). \quad (11)$$

Therefore, the exponents are described by the two independent parameters, z_{RW} and d_{f}. The dynamic exponent of random walks on a fractal substrate is given as $z_{\mathrm{RW}}=\frac{2d_{\mathrm{f}}}{d_s}$, where d_s is the spectral dimension defined by the density of normal modes on fractal lattices via $\rho(\omega)\sim w^{d_s-1}$. Thus, the results can be described by both d_s and d_{f} as

$$\beta=\frac{1}{2}-\frac{d_s}{4},\quad \alpha=d_{\mathrm{f}}\left(\frac{1}{d_s}-\frac{1}{2}\right),\quad z=\frac{2d_{\mathrm{f}}}{d_s}. \quad (12)$$

With these results, the scaling relation

$$2\alpha+d_{\mathrm{f}}=z \quad (13)$$

is obtained for the fractional Langevin equation. It is, thus, interesting to examine these results with respect to the ERSOS model on various fractal substrates.

The results of the exponents β and α, measured by direct Monte Carlo simulations, are presented in the following sections with respect to the ERSOS growth model on the Sierpinski gasket, checkerboard fractal, and Sierpinski carpet substrates, in conjunction with the scaling analysis of the surface width, in order to confirm the exponents.

III. BOUNDARY CONDITIONS OF FRACTAL SUBSTRATES

In this section, we describe the boundary conditions of the selected fractal lattices, utilized for the growth using the ER-SOS model. We particularly select three typical fractal substrates regarding the growth, i.e., the Sierpinski gasket, checkerboard fractal, and Sierpiski carpet, as shown in Fig. 1. The construction of these fractals are well described in Ref. [3], and the iterative growing processes are utilized. The fractal dimensions are $d_f^{sg}=\frac{\ln 3}{\ln 2}$, $d_f^{cb}=\frac{\ln 5}{\ln 3}$, and $d_f^{sc}=\frac{\ln 8}{\ln 3}$ with respect to the Sierpinksi gasket, checkerboard fractal, and Sierpinski carpet, respectively.

The lattice sites are defined on the vertices of the base triangles for the Sierpinski gasket and on the centers of the base squares for the checkerboard and Sierpnski carpet. The growth proceeds on the lattice sites of the fractal substrate and, accordingly, the height variable is defined on those lattice sites. When a particle is deposited or evaporated at the corner site on the edge, the nearest-neighbor sites to be examined for the RSOS condition are limited due to the system boundary. In order to reduce the size effect, it is necessary to set up a certain boundary condition. In regards to the usual critical phenomena, the two boundary conditions (such as the periodic boundary and the free boundary conditions) are employed, and the former is known to be more successful in reducing the size effect. Setting up the periodic boundaries for a checkerboard fractal is rather simple, as can be seen later. In regards to the Sierpinski gasket, on the other hand, since the translational invariance is not satisfied, it is not simple to employ the periodic boundary condition.

Considering that the Sierpinski gasket *ABC* in Fig. 1 is the current cell which has been generated up to the third order and assuming that the current cell is the subcell of a higher-order generation, the three replicated cells can be assumed to be at the three corner sites, part of which are as shown with the thin lines. Since the given cell and its replicated cells are of the same structure, the site B may be considered to be the replicated site of site A in the replicated cell of the right-hand side. Therefore, when the deposition of a particle is considered on the site B, the RSOS condition may be examined regarding the two nearest-neighbor sites of the site B and two of the site A. The same is applied for the site C. However, when a deposition on the site A is considered, the RSOS condition must be examined regarding all the nearest-neighbor sites of the sites A, B, and C, because A is the corresponding site to the replicated sites B and C. In order for this condition to be satisfied, the sites A, B, and C should be considered as effectively the same site; i.e., when any of these three sites are selected for deposition, the RSOS condition must be examined regarding the six neighboring sites. This boundary condition is referred to as the "periodic" boundary condition. In this condition, however, growth on the corner site may be suppressed due to the extra restrictions, compared to the growth on other sites, for which the coordination number is four. It is, however, believed based on the universality concept that the additional restriction on the boundary sites is irrelevant; i.e., it does not alter the critical behavior. In order to confirm this postulate, the "free" or "reflective" boundary condition is also utilized. With the

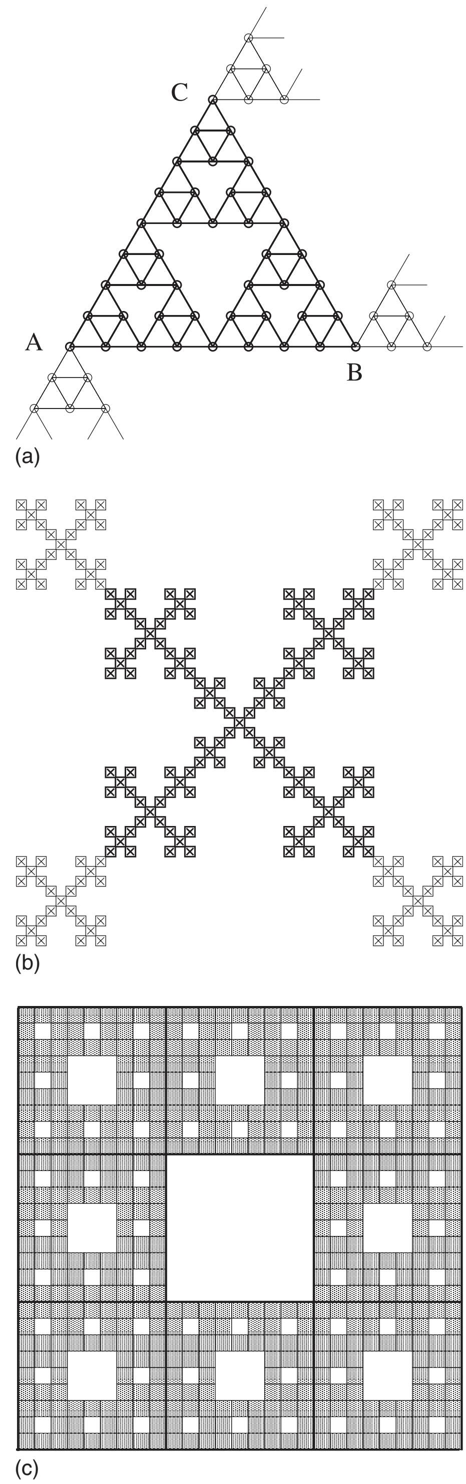

FIG. 1. A Sierpinski gasket (top) and a checkerboard fractal (middle), generated up to the third order (marked as thick lines), with parts of the periodically placed replicated cells (marked as thin lines), and a Sierpinski carpet (bottom). The lattice sites are set at the center of each square with respect to the checkerboard fractal as well as a Sierpinski carpet and at the vertices of the triangles with respect to the Sierpinski gasket.

free boundary condition, the RSOS condition is examined for all corner sites with the two nearest-neighbor sites in the given cell; therefore, growth on the boundary site may be

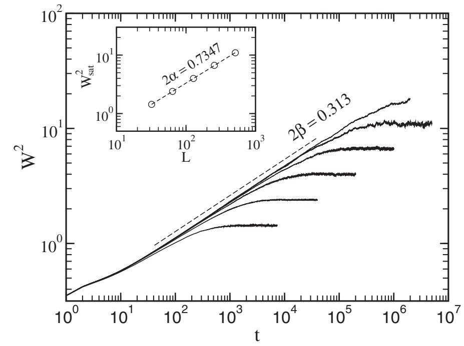

FIG. 2. Mean-square surface width W^2 as a function of time plotted on a double-logarithmic scale for the equilibrium RSOS model on a Sierpinski gasket substrate generated up to, from bottom to top, the fifth, sixth, seventh, eighth, ninth, and tenth generations.

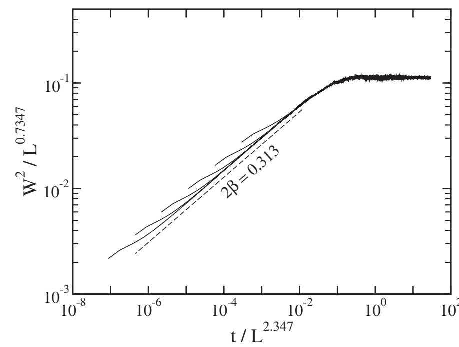

FIG. 3. Plot of the scaled mean-square surface width, W^2, as a function of the scaled time on a log-log scale for the equilibrium model on a Sierpinski gasket.

elevated slightly due to a lesser number of restrictions. We find that the two boundary conditions yield the same exponents.

In regards to the checkerboard fractal, the current cell may be assumed as the central cell of a higher-order generation, and the four replicated cells are assumed to be at the four corners of the given cell. When a deposition on the corner site is considered, the RSOS condition is examined for one nearest-neighbor site of the selected corner site and one nearest-neighbor site opposite to the selected site along the diagonal direction. Therefore, in this case, the number of nearest-neighbor sites to be examined for the RSOS condition is two regarding the four boundary sites in the system.

In regards to the Sierpinski carpet, neither the method applied for the Sierpinski gasket nor that of the checkerboard fractal can be applicable because the Sierpinski carpet is intrinsically different from the other two fractal lattices. While the two lattices are finitely ramified fractals, the Sierpinski carpet is an infinitely ramified fractal, i.e., the number of boundary sites increases as the size of fractal increases. For this reason, we attempt to employ the free boundary condition.

IV. NUMERICAL RESULTS

Simulations of the ERSOS growth are performed on fractal substrates with respect to the linear sizes up to $L=2^{10}$ for a Sierpinski gasket, up to $L=3^6$ for a checkerboard fractal, and up to $L=3^5$ for a Sierpinski carpet.

A. On a Sierpinski gasket

The mean-square surface width is calculated on a Sierpinski gasket regarding the sizes of 2^5 up to 2^{10}, and the results are plotted in Fig. 2. Based on the regression fit of the data in the early time, we obtain $\beta=0.157\pm0.002$. In the long-time region, the saturated width follows a power-law $W(L)\sim L^{\alpha}$, and the exponent α is estimated from the data in the inset as $\alpha=0.367\pm0.002$. From the values of β and α, we obtain $z\approx2.35$, which is considerably different from the value on the regular substrate, $z=2$. The scaled mean-square surface width, $W^2/L^{2\alpha}$, plotted against the scaled time, t/L^z, yields a perfect data collapse, with the exception of the early-time data, as shown in Fig. 3.

For the random walks on fractal substrates, the dynamic exponent z_{RW} is identical to the fractal dimension d_w of random walks, defined by the mean-square end-to-end displacement of the t-step random walks, $\langle R^2\rangle\sim t^{2/d_w}$. For the Sierpinski gasket, $d_{\mathrm{f}}^{\mathrm{sg}}=\frac{\ln 3}{\ln 2}\simeq1.585$, $d_s^{\mathrm{sg}}=2\frac{\ln 3}{\ln 5}\simeq1.365$, and $d_w^{\mathrm{sg}}=\frac{2d_{\mathrm{f}}}{d_s}=\frac{\ln 5}{\ln 2}\simeq2.322$ [21]. Therefore, the predicted values from Eq. (12) are $\beta=\frac{1}{2}(1-\frac{\ln 3}{\ln 5})\simeq0.159$ and $\alpha=\frac{\ln 5-\ln 3}{2\ln 2}\simeq0.3685$, which are in complete agreement with the estimates obtained by the simulations. The dynamic exponent z is also consistent with $z_{\mathrm{RW}}\simeq2.322$ within a 2% error margin. Therefore, it is surmised that $z=z_{\mathrm{RW}}$ is generally true with respect to the equilibrium RSOS model.

B. On a checkerboard fractal

Simulations for the equilibrium RSOS model are also carried out on a checkerboard fractal substrate. The critical exponents obtained are $\beta=0.203\pm0.002$ and $\alpha=0.503\pm0.003$, as shown in Fig. 4. The dynamic exponent is obtained as $z=\frac{\alpha}{\beta}\simeq2.48$. The scaling plot, using the measured values of the exponents, is shown in Fig. 5. An excellent data collapse confirms the estimates.

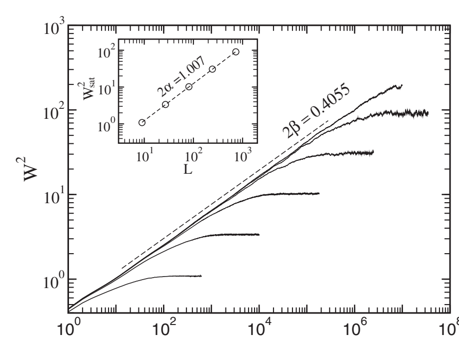

FIG. 4. Mean-square surface width W^2 as a function of time plotted on a double-logarithmic scale for the equilibrium model on a checkerboard fractal substrate. Data are, from bottom to top, for $L=3^2$, 3^3, 3^4, 3^5, 3^6, and 3^7. Plotted in the inset are the data for the saturated values against the size of the system.

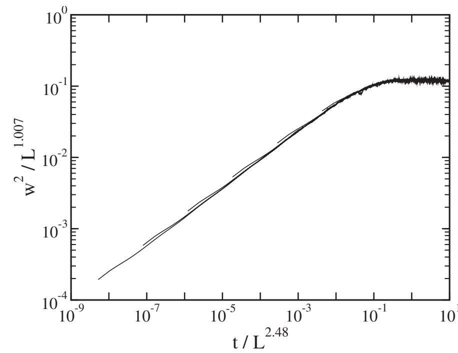

FIG. 5. Plot of the scaled mean-square surface width W^2 as a function of the scaled time on a log-log scale for the equilibrium model on a checkerboard fractal substrate.

The fractal dimension and the spectral dimension of a checkerboard fractal are, respectively, $d_f^{cb}=\frac{\ln 5}{\ln 3}\simeq 1.465$ and $d_s^{cb}=\frac{2\ln 5}{\ln 15}\simeq 1.189$ [22]. Therefore, the predicted values from Eq. (12) are $\beta=\frac{1}{2}-\frac{\ln 5}{2\ln 15}\simeq 0.203$ and $\alpha=\frac{\ln 5}{\ln 3}(\frac{\ln 15}{2\ln 5}-\frac{1}{2})=\frac{1}{2}$, which are in complete agreement with our estimates. The dynamic exponent, $z=d_w^{cb}=\frac{2d_f}{d_s}\simeq 2.465$, is also consistent with the given estimate within an error margin of less than 1%. Based on these results and those for the Sierpinski gasket, we conclude that the analytical predictions by a power-counting analysis is amazingly accurate and might even be exact.

C. On a Sierpinski carpet

Since $d_f=\frac{\ln 8}{\ln 3}\simeq 1.893$ and $d_s\simeq 1.802$ [23] for the Sierpinski carpet, the predicted values are obtained as $\beta\approx 0.050$ and $\alpha\approx 0.104$. The simulation results, however, yield a larger value for the exponent β and smaller value for α. The regression slope in the early time varies as the size of system increases, as shown in Fig. 6 and, for $L=3^5$, data yields the slope of about 0.18, corresponding to $\beta\simeq 0.09$. Considering the trend of the data, the true value of β appears to be smaller than this value. By extrapolating the measured values in the $L\to\infty$ limit, $\beta\approx 0.065$ is obtained, which is much closer to but is still slightly off from the predicted value. The value of α may be estimated, based on the data for $L=3^3$, 3^4, and 3^5, as $\alpha\simeq 0.142$, which is again deviated from the predicted value. Thus, the exponents for growth on a Sierpinski carpet were not in perfect agreement with the estimated values.

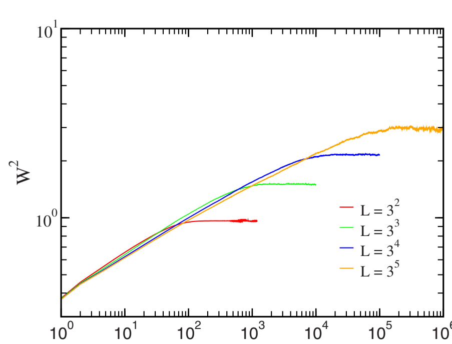

FIG. 6. (Color online) Mean-square surface width W^2 as a function of time plotted on a double-logarithmic scale for the equilibrium model on a Sierpinski carpet fractal substrate. Data are, from bottom to top, for $L=3^2$, 3^3, 3^4, and 3^5.

It is clear from Fig. 6 that the vertical displacements of the saturated values $W_{sat}^2(L)$ between the two neighboring sizes are not identical. This implies that the saturated width may not follow the power-law behavior of Eq. (7). It should be noted that, in order for the scaling to hold, the saturated values should be displaced with equal spacing for the two consecutive sizes, because the sizes are multiples of three. We believe that this is due to the finite-size effect and, if the size of system is further increased, data would yield consistent values, with a proper asymptotic growth exponent. The displacement between the systems of $L=3^3$ and 3^4 and that between the systems of $L=3^4$ and 3^5 are much smaller in comparison to that between the systems of $L=3^2$ and 3^3. This might be an indication that the inconsistent estimates were caused by the finite-size effect.

The finite-size effect may also be examined by considering the ratio of the number of boundary sites, i.e., the sites that are neighbors to the voids, to the total number of sites at the kth generation. The ratio decreases as 0.938, 0.805, 0.741, 0.716, 0.706, and 0.702, as k increases from $k=2$ to $k=7$. Plotting these values against $1/k$, it is found that the ratio decreases rapidly for small k and appears to approach to a constant between 0.69 and 0.7 as $k\to\infty$. If this ratio becomes smaller, i.e., if more sites with four neighbors exist, the growth will be suppressed because the RSOS condition should be examined over more neighboring sites. The smaller slopes for the larger systems in Fig. 6 are consistent with this expectation. Therefore, in order for the scaling relation in Eq. (7) to hold, this ratio must be constant and the size of system must be essentially infinite. (Note that this ratio is 0.2 with respect to the checkerboard fractal, irrespective of the size of systems.) However, considering the convergence behavior of the ratio against $1/k$, it is expected that approximate scaling may be observable for systems of $k\geq 7$. Since simulations with respect to the systems of size $L=3^6$, up to 10^7 time steps are estimated to take more than several months with an Intel 3.0GHz CPU, even for a single trial, such simulations are considered to be impossible with the present systems.

Simulations are also carried out using the periodic boundary condition, assuming that the current system is the lower-center subcell of a higher-order generation. In this boundary condition, when a particle is deposited on the right or left edge sites, the RSOS condition is examined between the site and its neighboring sites in opposite edges. The results were found to be basically similar to those in Fig. 6 (not shown). We also consider the ratio of the boundary area to the volume. It quickly approaches to zero with the system size L following L^{-d_f} in both the Sierpinski gasket and the checker board fractals. However, it decreases slowly as $L^{-(d_f-1)}$ for the Sierpinski carpet of the infinitely ramified fractal. This could explain the slow convergence of the exponents to the estimated values in the Sierpinski carpet.

V. SUMMARY AND CONCLUSIONS

We have studied the surface fluctuations of the equilibrium RSOS model on a fractal substrate by introducing a fractional Langevin equation to describe the model. The growth exponent, the roughness exponent, and the dynamic exponent were obtained by a power-counting analysis. The critical exponents depended upon the intrinsic properties of the underlying fractal substrates via the fractal dimension and the spectral dimension, and the dynamic exponent was the same as that for the random walks. In addition, we obtained the scaling relation $2\alpha + d_f = z$ regarding the ERSOS model on fractal substrates.

The results were examined numerically via Monte Carlo simulations regarding the equilibrium growth on a Sierpinski gasket, checkerboard fractal, and Sierpinski carpet. It was found that the results regarding the former two fractal substrates were consistent with the analytical predictions within a margin of error less than 2%. The results for the Sierpinski carpet were less successful, presumably due to the finite-size effect, which might be significant for infinitely ramified fractal lattices. Therefore, in order to derive a concrete conclusion on the validity of the predictions with respect to the Sierpinski carpet substrate, the results regarding the larger systems is necessary.

The results were summarized in Table I, together with the results on a critical percolation network taken from Ref. [24], as well as with those on a three-dimensional (3D) Sierpinski gasket substrate from Ref. [20]. A critical percolation network is known to be fractal with $d_f = \frac{91}{48}$ and $d_s \simeq 1.31$ [21] in two dimensions. The agreements between the predicted values and the simulation results were excellent. This confirms that the prediction in Eq. (12) is also successful, even with respect to random fractals. In regards to the growth on a Sierpinski gasket embedded in three dimensions, $d_s^{sg} = \frac{4 \ln 2}{\ln 6} \simeq 1.547$, and $d_f = 2$; thus, it is expected that $\beta \approx 0.113$ and $\alpha \approx 0.292$, based on Eq. (12). In the previous study, $\beta = 0.110 \pm 0.005$ and $\alpha = 0.293 \pm 0.005$ [20] have been obtained, which are in complete agreement with the predictions.

TABLE I. The critical exponents of equilibrium RSOS models on various fractal substrates.

Models	β	α	z
2D Sierpinski gasket:			
Monte Carlo	0.157 ± 0.002	0.367 ± 0.002	2.35
Equation (12)	0.1587	0.3685	2.322
2D Checkerboard:			
Monte Carlo	0.203 ± 0.002	0.504 ± 0.003	2.48
Equation (12)	0.2028	0.5	2.465
2D Percolation:			
Monte Carlo[a]	0.176 ± 0.001	0.509 ± 0.006	2.89
Equation (12)	0.175	0.510	2.89
3D Sierpinski gasket:			
Monte Carlo[b]	0.110 ± 0.005	0.293 ± 0.005	2.66
Equation (12)	0.1132	0.2925	2.585

[a]Reference [24].
[b]Reference [20].

It is interesting to note that the results in Eq. (12) were the same as those obtained by Zumofen *et al.*, who examined the scaling theory with respected to the surface width for the EW equation on a fractal substrate by means of a different approach, in which the behavior of the autocorrelation function of the associated diffusion problem had been studied [25], considering that the diffusion term $\nabla^2 h$ of the EW equation on fractal substrate can be written as $\Sigma_\delta (h_{i+\delta} - h_i)$ with respect to the nearest neighbors [26].

In regards to the nonequilibrium RSOS growth model, however, a similar approach was not successful, due to the nonlinear term in the KPZ equation. A deeper understanding is required and such a study is currently ongoing.

ACKNOWLEDGMENTS

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